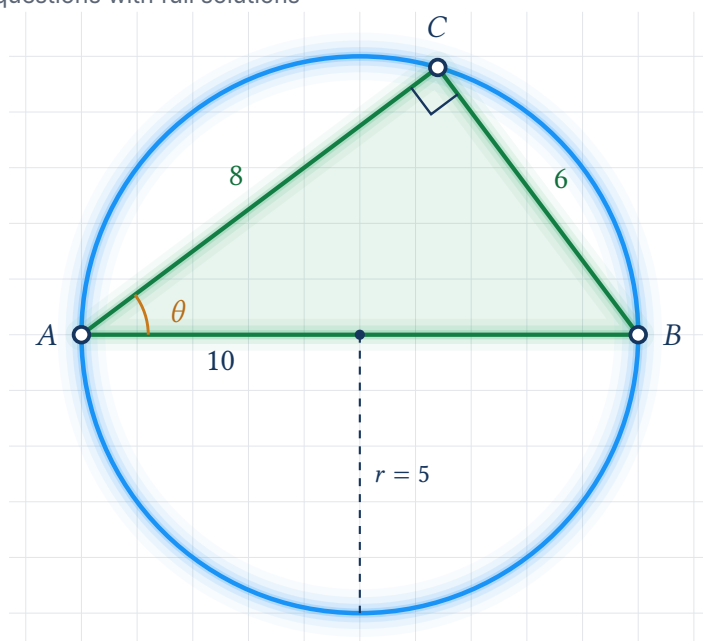


# Geometry & Trigonometry

## Unit 4 Study Notes

Key ideas · Worked examples · Desmos tips  
15 practice questions with full solutions



$$\sin \theta = \frac{6}{10} = \frac{3}{5}$$

$$\cos \theta = \frac{8}{10} = \frac{4}{5}$$

angle  $C = 90^\circ$ :  
the hypotenuse  $AB$   
is a diameter

### What these notes cover

Geometry and Trigonometry is about 15% of SAT Math, roughly 5–7 of the 44 questions. Many formulas are on the reference sheet in the test; the skill is knowing which one to use. The College Board lists four skills:

- 1 Lines, angles and triangles
- 2 Area and volume
- 3 Right triangles and trigonometry
- 4 Circles

### How to use them

Read a section, cover the solution to each worked example and try it yourself first. Then do the 15 practice questions in one sitting, with Desmos open, and check your work against the full solutions at the end.

## Contents

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## How the SAT tests this unit

There are only a few Geometry questions, but they reward a short list of facts known cold. Diagrams are not always drawn to scale, so trust the numbers and the given facts, not your eye.

| Skill                                | What the questions actually ask                                                                                                    |
|--------------------------------------|------------------------------------------------------------------------------------------------------------------------------------|
| <b>Lines, angles, triangles</b>      | Angles at parallel lines; angle sums; isosceles triangles; similar triangles and their proportional sides.                         |
| <b>Area and volume</b>               | Use the reference formulas; work backwards from a given area or volume; see how area and volume change when lengths are scaled.    |
| <b>Right triangles, trigonometry</b> | Pythagorean theorem; the 30-60-90 and 45-45-90 triangles; sine, cosine and tangent; $\sin x^\circ = \cos(90 - x)^\circ$ ; radians. |
| <b>Circles</b>                       | Arc length and sector area; the equation of a circle, including completing the square to find its center and radius.               |

### THE BIG IDEA OF THIS UNIT

**Mark the figure.** Write every given length and angle on the diagram, then add what you can deduce: equal angles, right angles, equal radii. Most Geometry questions are solved once the figure is fully labeled.

## 1 Lines, angles and triangles

### ANGLE FACTS

Angles on a straight line add to  $180^\circ$ ; vertical angles are equal.

When a line crosses two **parallel** lines, *corresponding* and *alternate interior* angles are equal, and *same-side interior* angles add to  $180^\circ$ .

The angles of a triangle add to  $180^\circ$ . An exterior angle equals the sum of the two opposite interior angles.

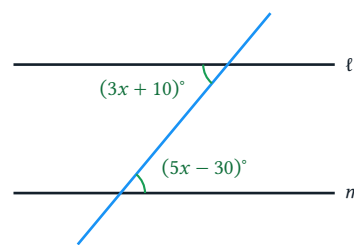


Figure 1.  $l \parallel m$ : alternate interior angles.

### WORKED EXAMPLE 1.1

In Figure 1, lines  $l$  and  $m$  are parallel. The marked angles are  $(3x + 10)^\circ$  and  $(5x - 30)^\circ$ . Find  $x$  and the size of the angles.

**Solution.** They are alternate interior angles, so they are equal:  $3x + 10 = 5x - 30$ , giving  $2x = 40$  and  $x = 20$ . Each angle is  $3(20) + 10 = 70^\circ$ .

## Similar triangles

Two triangles are **similar** when their angles match. Then all their sides are in the same ratio, and you can set up a proportion between matching sides.

### WORKED EXAMPLE 1.2

In triangle  $ABC$ , point  $D$  is on  $AB$  and point  $E$  is on  $AC$ , with  $DE$  parallel to  $BC$ . If  $AD = 4$ ,  $DB = 6$  and  $DE = 5$ , find  $BC$ .

**Solution.** Because  $DE \parallel BC$ , triangle  $ADE$  is similar to triangle  $ABC$ . Matching sides:  $\frac{BC}{DE} = \frac{AB}{AD} = \frac{4 + 6}{4}$ , so  $BC = 5 \times \frac{10}{4} = 12.5$ .

### SAT TRAP

Match the *corresponding* sides, the ones opposite equal angles, not the sides that look alike. In Example 1.2 the whole side  $AB = 10$  matches  $AD = 4$ ; using  $DB = 6$  instead is the classic mistake.

## 2 Area and volume

### FORMULAS TO KNOW, AND HOW SCALING WORKS

Rectangle, triangle  $A = \ell w$ ,  $A = \frac{1}{2}bh$

Circle  $A = \pi r^2$ , circumference  $C = 2\pi r$

Box, cylinder  $V = \ell wh$ ,  $V = \pi r^2 h$

Cone, sphere  $V = \frac{1}{3}\pi r^2 h$ ,  $V = \frac{4}{3}\pi r^3$

Scaling If every length is multiplied by  $k$ , every area is multiplied by  $k^2$  and every volume by  $k^3$ .

### WORKED EXAMPLE 2.1

A cylinder has radius 3 and height 10. Find its volume.

**Solution.**  $V = \pi r^2 h = \pi(3)^2(10) = 90\pi$ .

### WORKED EXAMPLE 2.2

A box measures 4 by 5 by 6. Each dimension is doubled. What happens to the volume?

**Solution.** The volume was  $4 \cdot 5 \cdot 6 = 120$  and becomes  $8 \cdot 10 \cdot 12 = 960$ : eight times as large, because  $2^3 = 8$ .

### WORKED EXAMPLE 2.3

A shape is a 10-by-6 rectangle with a semicircle on one of its 6-unit sides. Find its area.

**Solution.** The semicircle has diameter 6, so radius 3. Area  $= 10 \times 6 + \frac{1}{2}\pi(3)^2 = 60 + 4.5\pi$ .

## 3 Right triangles and trigonometry

### THE RIGHT-TRIANGLE TOOLKIT

Pythagorean theorem:  $a^2 + b^2 = c^2$ , where  $c$  is the hypotenuse. Common triples: 3, 4, 5; 5, 12, 13; 8, 15, 17, and their multiples.

Special triangles:  $45^\circ$ - $45^\circ$ - $90^\circ$  has sides  $x$ ,  $x$ ,  $x\sqrt{2}$ ;  $30^\circ$ - $60^\circ$ - $90^\circ$  has sides  $x$ ,  $x\sqrt{3}$ ,  $2x$ .

SOH CAH TOA:  $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$ ,  $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ ,  $\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$ .

Complementary angles:  $\sin x^\circ = \cos(90 - x)^\circ$ . Radians:  $\pi$  radians  $= 180^\circ$ .

### WORKED EXAMPLE 3.1

A right triangle has legs 5 and 12. Find the hypotenuse.

**Solution.**  $c = \sqrt{25 + 144} = \sqrt{169} = 13$ , the 5, 12, 13 triple.

**WORKED EXAMPLE 3.2**

In a right triangle, one angle is  $30^\circ$  and the hypotenuse is 10. Find the other two sides.

**Solution.** The side opposite  $30^\circ$  is half the hypotenuse: 5. The side opposite  $60^\circ$  is  $5\sqrt{3}$ .

**WORKED EXAMPLE 3.3**

In the triangle on the cover,  $\sin \theta = \frac{3}{5}$ . Find  $\cos \theta$  and  $\tan \theta$ .

**Solution.** Opposite 6, hypotenuse 10, so the adjacent side is 8. Then  $\cos \theta = \frac{8}{10} = \frac{4}{5}$  and  $\tan \theta = \frac{6}{8} = \frac{3}{4}$ . You can always build a triangle from one trig ratio this way.

**WORKED EXAMPLE 3.4** HARD · MODULE 2

If  $\sin x^\circ = \cos 40^\circ$  and  $0 < x < 90$ , what is  $x$ ?

**Solution.** The sine of an angle equals the cosine of its complement, so  $x = 90 - 40 = 50$ .

**SAT TRAP**

Convert between degrees and radians with  $\pi = 180^\circ$ : for example  $\frac{3\pi}{4}$  radians  $= \frac{3}{4} \times 180^\circ = 135^\circ$ . And in Desmos, check the angle mode (degrees or radians) before you compute any sine or cosine.

**4 Circles****CIRCLE FACTS**

**Equation:**  $(x - h)^2 + (y - k)^2 = r^2$  is a circle with center  $(h, k)$  and radius  $r$ .

**Arc length**  $= \frac{\theta}{360^\circ} \times 2\pi r$  and **sector area**  $= \frac{\theta}{360^\circ} \times \pi r^2$ , for a central angle  $\theta$  in degrees. In radians, arc length  $= r\theta$ .

A tangent line is perpendicular to the radius at the point where it touches. An inscribed angle is half the central angle on the same arc, so an angle inscribed in a semicircle is  $90^\circ$  (as on the cover).

**WORKED EXAMPLE 4.1** HARD · MODULE 2

Find the center and radius of the circle  $x^2 + y^2 - 6x + 4y - 12 = 0$ .

**Solution.** Complete the square in  $x$  and in  $y$ :

$$(x^2 - 6x + 9) + (y^2 + 4y + 4) = 12 + 9 + 4 \implies (x - 3)^2 + (y + 2)^2 = 25.$$

The center is  $(3, -2)$  and the radius is  $\sqrt{25} = 5$ . Watch the signs:  $(y + 2)^2$  means  $k = -2$ .

**WORKED EXAMPLE 4.2**

A circle has radius 9. Find the length of the arc cut off by a central angle of  $80^\circ$ .

**Solution.**  $\frac{80}{360} \times 2\pi(9) = \frac{2}{9} \times 18\pi = 4\pi$ .

**SAT TRAP**

In the equation of a circle the right side is  $r^2$ , not  $r$ . For  $(x - 3)^2 + (y + 2)^2 = 25$  the radius is 5, not 25.

## 5 Your Desmos toolkit for Geometry

Geometry is the unit where Desmos helps least with the reasoning, but it is excellent for circles and for checking any calculation with trigonometry.

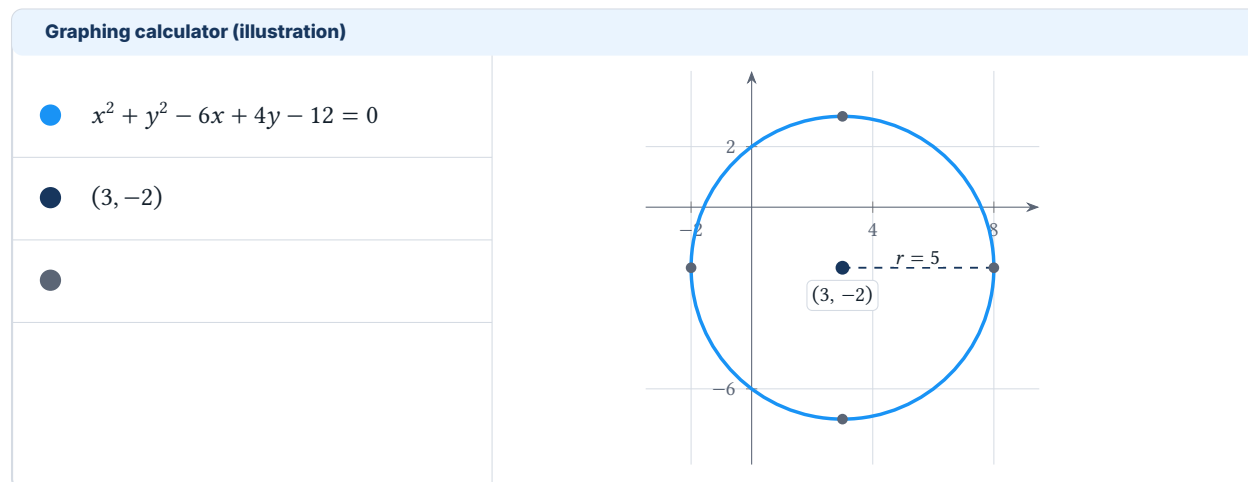


Figure 2. Type the equation exactly as given: Desmos draws the circle, and the marked points give the center and radius.

### Five Desmos moves for Geometry

#### 1 · CIRCLES WITHOUT COMPLETING THE SQUARE

Type the circle's equation in any form. Click the extreme left and right points: halfway between them is the center, and half the distance is the radius.

#### 2 · CHECK THE ANGLE MODE

Open the settings (wrench) and choose Degrees or Radians before any trigonometry. Then  $\sin(30)$  gives 0.5 in degree mode.

#### 3 · PYTHAGORAS IN ONE LINE

Type  $\text{sqrt}(5^2 + 12^2)$  to get 13. For a missing leg, type  $\text{sqrt}(13^2 - 5^2)$ .

#### 4 · FIND AN ANGLE FROM A RATIO

Type  $\arcsin(3/5)$  in degree mode to get the angle, about  $36.87^\circ$ . The same works with  $\arccos$  and  $\arctan$ .

#### 5 · LINE MEETS CIRCLE

Type the circle and the line; click the intersections to read the points. A line that only touches the circle once is tangent to it.

#### SAT TRAP

Desmos decimals hide exact forms. If the choices are  $4\pi$  or  $5\sqrt{3}$ , compare with  $4\pi$  or  $5\text{sqrt}(3)$  typed into Desmos, rather than guessing from the decimal.

**6 Practice questions**

Fifteen original questions in the SAT style, arranged from easy to hard. Aim to finish in about 20 minutes, with Desmos open. Questions marked **HARD · MODULE 2** are the kind that separate good scores from top scores.

**1 EASY**

Two parallel lines are cut by a transversal. One of two same-side interior angles measures  $65^\circ$ . What is the measure, in degrees, of the other?

Student-produced response: enter your answer.

**2 EASY**

Two angles of a triangle measure  $50^\circ$  and  $60^\circ$ . What is the measure of the third angle?

- A**  $60^\circ$  **B**  $70^\circ$   
**C**  $80^\circ$  **D**  $110^\circ$

**3 EASY**

A right triangle has legs of length 6 and 8. What is the length of the hypotenuse?

Student-produced response: enter your answer.

**4 EASY**

A rectangle has an area of 48 square inches and a width of 6 inches. What is its perimeter, in inches?

- A** 14 **B** 42  
**C** 28 **D** 48

**5 EASY**

What is the area of a circle with radius 7?

- A**  $7\pi$  **B**  $14\pi$   
**C**  $196\pi$  **D**  $49\pi$

**6 EASY**

Which of the following is equal to  $60^\circ$ ?

- A**  $\pi/6$  radians **B**  $\pi/3$  radians  
**C**  $\pi/2$  radians **D**  $2\pi/3$  radians

**7 MEDIUM**

A cylinder has radius 4 and height 5. Its volume is  $k\pi$ . What is the value of  $k$ ?

Student-produced response: enter your answer.

**8 MEDIUM**

A triangle with sides 3, 4 and 5 is similar to a second triangle whose shortest side is 12. What is the length of the longest side of the second triangle?

Student-produced response: enter your answer.

**9 MEDIUM**

In right triangle  $ABC$ , angle  $C$  is the right angle and  $\sin A = \frac{5}{13}$ . What is  $\cos A$ ?

- A**  $\frac{5}{12}$  **B**  $\frac{12}{13}$   
**C**  $\frac{12}{13}$  **D**  $\frac{12}{5}$

**10 MEDIUM**

Which describes the circle  $(x+1)^2 + (y-4)^2 = 36$ ?

- A** Center  $(-1, 4)$ , radius 6  
**B** Center  $(1, -4)$ , radius 6  
**C** Center  $(-1, 4)$ , radius 36  
**D** Center  $(1, -4)$ , radius 36

**11 MEDIUM**

The hypotenuse of a  $45^\circ$ - $45^\circ$ - $90^\circ$  triangle is  $10\sqrt{2}$ . What is the length of each leg?

Student-produced response: enter your answer.

**12 MEDIUM**

A circle has radius 12. What is the length of the arc cut off by a central angle of  $60^\circ$ ?

- A**  $2\pi$  **B**  $4\pi$   
**C**  $12\pi$  **D**  $24\pi$

**13** HARD · MODULE 2

If  $\sin x^\circ = \cos 32^\circ$  and  $0 < x < 90$ , what is the value of  $x$ ?

Student-produced response: enter your answer.

**14** HARD · MODULE 2

What is the radius of the circle  $x^2 + y^2 + 8x - 6y = 0$ ?

Student-produced response: enter your answer.

**15** HARD · MODULE 2

A cube has a volume of 64 cubic inches. Each edge is increased by 50%. The new volume is how many times the original volume?

- A** 1.5                      **B** 2.25  
**C** 3                        **D** 3.375

**LEARN FROM EVERY MISTAKE**

Before reading the solutions, mark each question you missed with the reason. Over time the pattern tells you exactly what to practice.

| Reason          | What it means                                                        | What to do                 |
|-----------------|----------------------------------------------------------------------|----------------------------|
| <b>Concept</b>  | I did not know the fact or formula.                                  | Re-read the section.       |
| <b>Method</b>   | I knew the idea but chose a slow or wrong approach.                  | Study the worked examples. |
| <b>Misread</b>  | I answered a different question (for example $r^2$ instead of $r$ ). | Mark the figure first.     |
| <b>Careless</b> | An arithmetic or sign slip.                                          | Check with Desmos.         |

**7 Full solutions****1 115**Same-side interior angles add to  $180^\circ$ :  $180 - 65 = 115$ .**2 B** $180 - 50 - 60 = 70$ .**3 10** $\sqrt{6^2 + 8^2} = \sqrt{100} = 10$ : the 3, 4, 5 triple doubled.**4 C**The length is  $48 \div 6 = 8$ , so the perimeter is  $2(8+6) = 28$ . Choice D, 48, is the area.**5 D** $\pi(7)^2 = 49\pi$ . Choice B,  $14\pi$ , is the circumference.**6 B**

$$60 \times \frac{\pi}{180} = \frac{\pi}{3}.$$

**7 80** $V = \pi(4)^2(5) = 80\pi$ , so  $k = 80$ .**8 20**The scale factor is  $12 \div 3 = 4$ , so the longest side is  $5 \times 4 = 20$ .**9 B**Build the triangle: opposite 5, hypotenuse 13, so the adjacent side is  $\sqrt{169 - 25} = 12$  and  $\cos A = \frac{12}{13}$ .**10 A** $(x - h)^2$  with  $h = -1$  and  $(y - k)^2$  with  $k = 4$ : center  $(-1, 4)$ . The radius is  $\sqrt{36} = 6$ .**11 10**The hypotenuse is a leg times  $\sqrt{2}$ , so each leg is  $10\sqrt{2} \div \sqrt{2} = 10$ .**12 B**

$$\frac{60}{360} \times 2\pi(12) = \frac{1}{6} \times 24\pi = 4\pi.$$

**13 58**Sine of an angle equals the cosine of its complement:  $x = 90 - 32 = 58$ .**14 5**Complete the square:  $(x^2 + 8x + 16) + (y^2 - 6y + 9) = 16 + 9$ , so  $(x + 4)^2 + (y - 3)^2 = 25$  and  $r = 5$ .**15 D**Each edge is multiplied by 1.5, so the volume is multiplied by  $1.5^3 = 3.375$  (from 64 to 216). Choice A forgets that volume has three dimensions; B is the area factor.**Want to go further?**

These notes are a sample of how I teach. In my live online SAT Math classes we work through all four units (Algebra, Advanced Math, Problem-Solving & Data Analysis, and Geometry & Trigonometry) in small groups, with full-length practice tests reviewed in class.

**Book a free demo class:** [sat@joelsamuelacademy.in](mailto:sat@joelsamuelacademy.in)

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