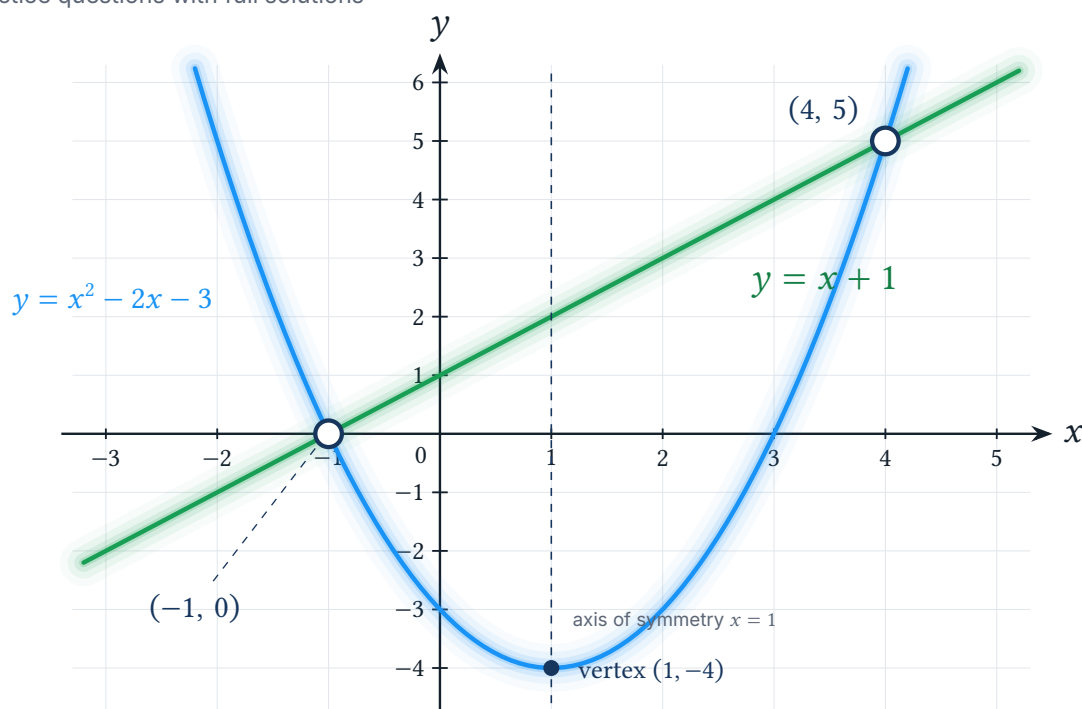


Advanced Math

Unit 2 Study Notes

Key ideas • Worked examples • Desmos tips
15 practice questions with full solutions



What these notes cover

Advanced Math is about 35% of SAT Math, roughly 13–15 of the 44 questions. It is the algebra of curves: anything that is *not* a straight line. The College Board groups it into three skills:

- 1 Equivalent expressions
- 2 Nonlinear equations in one variable and systems of equations in two variables
- 3 Nonlinear functions (quadratic and exponential)

How to use them

Read a section, cover the solution to each worked example and try it yourself first. Then do the 15 practice questions in one sitting, with Desmos open, and check your work against the full solutions at the end.

Contents

1	Equivalent expressions	3
2	Quadratic and other nonlinear equations	4
3	Nonlinear functions: parabolas and exponentials	5
4	Systems with a curve and a line	6
5	Your Desmos toolkit for Advanced Math	7
6	Practice questions: 6 easy, 6 medium, 3 hard	8
7	Full solutions	10

How the SAT tests Advanced Math

The questions rarely stop at “solve”. They ask you to see a structure: which form of a quadratic shows the vertex, what the base of an exponential says about growth, or how many times a line meets a curve.

Skill	What the questions actually ask
Equivalent expressions	Expand, factor and simplify; use exponent rules; rewrite roots as fractional powers; cancel common factors in fractions.
Nonlinear equations	Solve quadratics by factoring or the formula; decide how many real solutions there are; spot extraneous solutions of radical equations.
Systems with a curve	Find where a line meets a parabola; choose a constant so they meet once, twice or never.
Nonlinear functions	Read the vertex, zeros and y -intercept of a parabola; interpret $a \cdot b^x$ as growth or decay; use function notation and shifts.

THE BIG IDEA OF THIS UNIT

The **form** of an expression tells you what it shows. Factored form shows the zeros, vertex form shows the turning point, and $a \cdot b^x$ shows the starting value and the growth factor. Much of Advanced Math is choosing, or rewriting into, the form that answers the question.

1 Equivalent expressions

Two expressions are *equivalent* if they give the same value for every x . The SAT tests the tools that turn one form into another.

THE TOOLS

Special products	$(a + b)^2 = a^2 + 2ab + b^2$, $(a - b)^2 = a^2 - 2ab + b^2$, $(a + b)(a - b) = a^2 - b^2$
Exponent rules	$x^m \cdot x^n = x^{m+n}$, $\frac{x^m}{x^n} = x^{m-n}$, $(x^m)^n = x^{mn}$, $x^{-n} = \frac{1}{x^n}$
Roots as powers	$\sqrt[n]{x^m} = x^{m/n}$, so $\sqrt{x} = x^{1/2}$ and $\sqrt[3]{x^2} = x^{2/3}$

WORKED EXAMPLE 1.1

Which expression is equivalent to $(2x - 3)^2 - (x - 1)(x + 1)$?

Solution. $(2x - 3)^2 = 4x^2 - 12x + 9$ and $(x - 1)(x + 1) = x^2 - 1$. Subtract, keeping the bracket:
 $4x^2 - 12x + 9 - x^2 + 1 = 3x^2 - 12x + 10$.

WORKED EXAMPLE 1.2

Write $\sqrt[3]{x^2} \cdot x$ as a single power of x , for $x > 0$.

Solution. $\sqrt[3]{x^2} = x^{2/3}$ and $x = x^1$. Add the exponents: $x^{2/3+1} = x^{5/3}$.

WORKED EXAMPLE 1.3

Simplify $\frac{x^2 - 9}{x^2 + 5x + 6}$, for $x \neq -2$ and $x \neq -3$.

Solution. Factor the top and the bottom, then cancel the common factor:

$$\frac{(x - 3)(x + 3)}{(x + 2)(x + 3)} = \frac{x - 3}{x + 2}.$$

SAT TRAP

$(a + b)^2$ is **not** $a^2 + b^2$: the middle term $2ab$ is the most common thing to forget. And you may only cancel *factors*, never terms: in $\frac{x + 3}{x + 5}$ nothing cancels.

2 Quadratic and other nonlinear equations

A quadratic equation can be written as $ax^2 + bx + c = 0$. There are three ways to solve it; pick the fastest for the numbers in front of you.

THREE METHODS

Factoring: if $(x - p)(x - q) = 0$, then $x = p$ or $x = q$.

Quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, which always works.

Completing the square: rewrite as $a(x - h)^2 + k$; this also gives the vertex of the graph (Section 3).

WORKED EXAMPLE 2.1

Solve $2x^2 - 7x + 3 = 0$.

Solution. Factor: $(2x - 1)(x - 3) = 0$, so $x = \frac{1}{2}$ or $x = 3$.

WORKED EXAMPLE 2.2

Solve $x^2 + 4x - 1 = 0$.

Solution. It does not factor nicely, so use the formula with $a = 1$, $b = 4$, $c = -1$:

$$x = \frac{-4 \pm \sqrt{16 + 4}}{2} = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}.$$

How many real solutions? The discriminant

The number under the square root, $b^2 - 4ac$, is the **discriminant**. It tells you how many real solutions there are, without solving:

Discriminant	What it means for the graph of $y = ax^2 + bx + c$	Real solutions
$b^2 - 4ac > 0$	The parabola crosses the x -axis twice.	two
$b^2 - 4ac = 0$	The vertex just touches the x -axis.	exactly one
$b^2 - 4ac < 0$	The parabola never reaches the x -axis.	none

WORKED EXAMPLE 2.3

For what value of c does $x^2 - 6x + c = 0$ have exactly one real solution?

Solution. Set the discriminant to zero: $(-6)^2 - 4(1)(c) = 0$, so $36 - 4c = 0$ and $c = 9$. (Then $x^2 - 6x + 9 = (x - 3)^2$.)

WORKED EXAMPLE 2.4 HARD · MODULE 2

The solutions of $3x^2 - 12x + 7 = 0$ are p and q . What is $p + q$?

Solution. For $ax^2 + bx + c = 0$, the sum of the solutions is $-\frac{b}{a}$ and their product is $\frac{c}{a}$. So $p + q = -\frac{-12}{3} = 4$, with no need to find p and q .

Radical equations: check for extraneous solutions**WORKED EXAMPLE 2.5**

Solve $\sqrt{x + 7} = x + 1$.

Solution. Square both sides: $x + 7 = x^2 + 2x + 1$, so $x^2 + x - 6 = 0$ and $(x + 3)(x - 2) = 0$. Now *check both* in the original equation: $x = 2$ gives $\sqrt{9} = 3$ ✓, but $x = -3$ gives $\sqrt{4} = 2$ on the left and -2 on the right. So $x = -3$ is **extraneous**, and the only solution is $x = 2$.

SAT TRAP

Squaring both sides can create false solutions, because a square root is never negative. Whenever you square to remove a root, substitute every answer back into the *original* equation.

3 Nonlinear functions: parabolas and exponentials**Reading a parabola**

The graph of a quadratic function is a **parabola**. If $a > 0$ it opens up and has a minimum; if $a < 0$ it opens down and has a maximum. Each form shows something different:

Standard $y = ax^2 + bx + c$: the y -intercept is c .

Factored $y = a(x - r)(x - s)$: the zeros are r and s .

Vertex $y = a(x - h)^2 + k$: the vertex is (h, k) .

The vertex lies on the axis of symmetry, exactly halfway between the zeros: $x = \frac{r + s}{2} = -\frac{b}{2a}$.

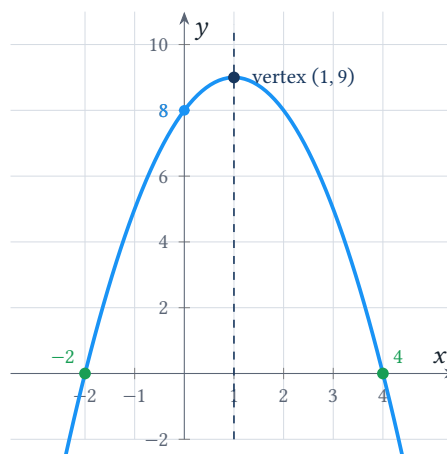


Figure 1. $y = -(x - 1)^2 + 9 = -(x - 4)(x + 2) = -x^2 + 2x + 8$:
three forms of one parabola.

WORKED EXAMPLE 3.1

For $f(x) = x^2 - 6x + 5$, find the minimum value of f .

Solution. Complete the square: $x^2 - 6x + 5 = (x - 3)^2 - 9 + 5 = (x - 3)^2 - 4$. The vertex is $(3, -4)$, so the minimum value is -4 (at $x = 3$).

Exponential growth and decay

An exponential function has the form $f(x) = a \cdot b^x$: a is the **starting value** and b is the **growth factor**, what you multiply by each step.

LINEAR ADDS, EXPONENTIAL MULTIPLIES

Linear	The <i>same amount</i> is added each step: $y = mx + b$.
Exponential	The <i>same factor</i> multiplies each step: $y = a \cdot b^x$.
Growth by $r\%$	$b = 1 + \frac{r}{100}$, so 3% growth means $b = 1.03$.
Decay by $r\%$	$b = 1 - \frac{r}{100}$, so a 15% loss means $b = 0.85$.

WORKED EXAMPLE 3.2

A car bought for \$24,000 loses 15% of its value each year. Write a model and find its value after 2 years.

Solution. $V(t) = 24,000(0.85)^t$. After 2 years: $24,000 \times 0.85^2 = 24,000 \times 0.7225 = \$17,340$.

WORKED EXAMPLE 3.3

If $f(x) = x^2 - 4x$, find $f(x + 1)$.

Solution. Replace *every* x by $(x + 1)$: $(x + 1)^2 - 4(x + 1) = x^2 + 2x + 1 - 4x - 4 = x^2 - 2x - 3$. Graph shifts work the same way: $f(x - 2) + 3$ is the graph of f moved 2 right and 3 up.

SAT TRAP

A quantity that doubles every 5 years is *not* $P_0 \cdot 2^t$. The exponent counts how many doubling periods have passed: $P_0 \cdot 2^{t/5}$. Read the time unit carefully.

4 Systems with a curve and a line

When one equation is a line and the other a parabola, substitute the line into the parabola. You get a quadratic, and its solutions are the x -coordinates of the meeting points.

WORKED EXAMPLE 4.1

Solve the system $y = x^2 - 2x - 3$ and $y = x + 1$ (the graphs on the cover).

Solution. Set the right sides equal: $x^2 - 2x - 3 = x + 1$, so $x^2 - 3x - 4 = 0$ and $(x - 4)(x + 1) = 0$. Then $x = 4$ gives $y = 5$, and $x = -1$ gives $y = 0$. The solutions are $(4, 5)$ and $(-1, 0)$.

WORKED EXAMPLE 4.2 HARD · MODULE 2

The line $y = kx$, where $k > 0$, meets the parabola $y = x^2 + 4$ at exactly one point. What is k ?

Solution. Substitute: $x^2 + 4 = kx$, so $x^2 - kx + 4 = 0$. Exactly one meeting point means exactly one solution, so the discriminant is zero: $k^2 - 16 = 0$, and since $k > 0$, $k = 4$. The line just touches (is tangent to) the parabola at $(2, 8)$.

5 Your Desmos toolkit for Advanced Math

Desmos is at its most powerful in this unit. Curves that take several lines of algebra to analyze can be read off a graph in seconds, and the graph is a reliable check on your algebra.

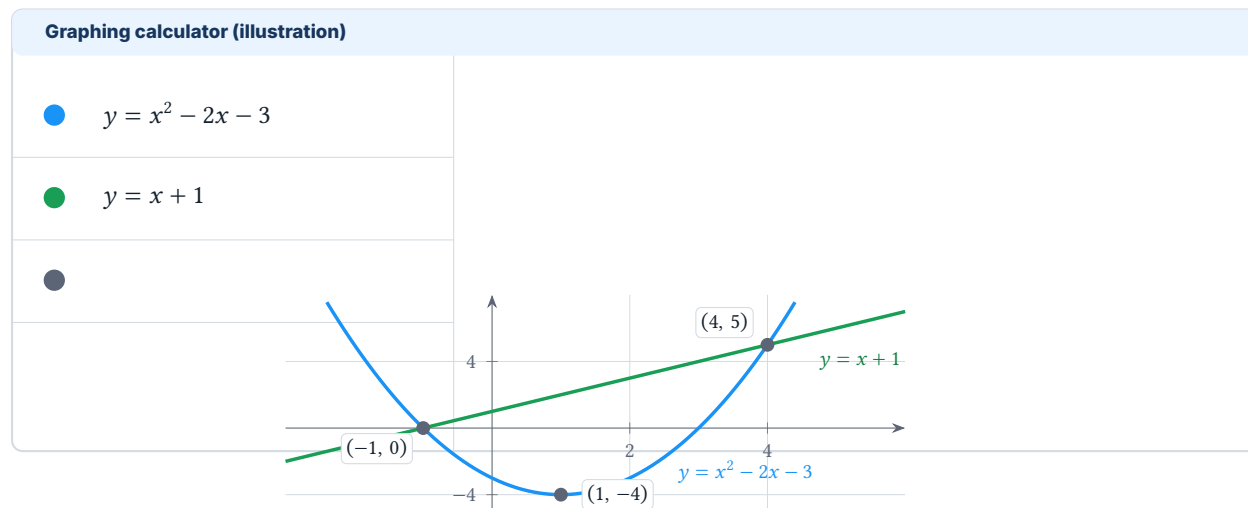


Figure 2. Click the curves and Desmos marks the meeting points $(-1, 0)$ and $(4, 5)$ and the vertex $(1, -4)$.

Five Desmos moves for Advanced Math

1 · ZEROS AND VERTEX IN ONE CLICK

Type the quadratic and click the curve. Desmos marks the x -intercepts, the y -intercept and the vertex as gray points; click one to read its coordinates.

2 · SOLVE ANY NONLINEAR EQUATION

Graph the left side and the right side as two curves. The x -coordinates of the meeting points are the solutions. This works for radical and exponential equations too.

3 · CHECK EQUIVALENT EXPRESSIONS

Graph the original expression and your answer. If you see only *one* curve, they are equivalent. Two separate curves mean a mistake somewhere.

4 · SLIDERS FOR “HOW MANY SOLUTIONS”?

Type $y = x^2 - 6x + c$; Desmos offers a slider for c . Drag it until the vertex just touches the x -axis: that is the one-solution case.

5 · READ VALUES FROM A MODEL

Type an exponential model such as $y = 24000(0.85)^x$, then use a table or click the curve at $x = 2$ to read the value directly.

SAT TRAP

Desmos only draws *real* meeting points, so an extraneous solution of a radical equation simply does not appear. That makes it a quick check after you square both sides.

6 Practice questions

Fifteen original questions in the SAT style, arranged from easy to hard. Aim to finish in about 20 minutes, with Desmos open. Questions marked **HARD · MODULE 2** are the kind that separate good scores from top scores.

1 EASY

Which expression is equivalent to $(x + 5)(x - 2)$?

- A** $x^2 - 3x - 10$ **B** $x^2 + 7x - 10$
C $x^2 + 3x - 10$ **D** $x^2 - 10$

2 EASY

If $x^2 = 49$ and $x < 0$, what is the value of x ?

Student-produced response: enter your answer.

3 EASY

If $f(x) = 2x^2 - 3$, what is the value of $f(-2)$?

Student-produced response: enter your answer.

4 EASY

For $x > 0$, which expression is equivalent to $x^{1/2} \cdot x^{3/2}$?

- A** x **B** x^2
C $x^{3/4}$ **D** x^3

5 EASY

A culture starts with 500 bacteria and the number doubles every hour. Which function gives the number of bacteria after h hours?

- A** $500 + 2h$ **B** $500(2)^h$
C $2(500)^h$ **D** $500h^2$

6 EASY

What are the solutions of $(x - 4)(2x + 6) = 0$?

- A** 4 and 3 **B** -4 and 3
C 4 and -6 **D** 4 and -3

7 MEDIUM

What is the sum of the solutions of $x^2 - 10x + 21 = 0$?

Student-produced response: enter your answer.

8 MEDIUM

The function f is defined by $f(x) = x^2 - 6x + 5$. What is the minimum value of $f(x)$?

Student-produced response: enter your answer.

9 MEDIUM

The value of a machine, in dollars, t years after it was bought is $V(t) = 18,000(0.9)^t$. By what percent does the value decrease each year?

- A** 0.9% **B** 9%
C 10% **D** 90%

10 MEDIUM

For $x \neq -4$, which expression is equivalent to $\frac{x^2 - 16}{x + 4}$?

- A** $x + 4$ **B** $x - 16$
C $x - 4$ **D** $x^2 - 4$

11 MEDIUM

What value of x satisfies $\sqrt{2x + 3} = x$?

Student-produced response: enter your answer.

12 MEDIUM

The graphs of $y = x^2$ and $y = 2x + 8$ meet at two points. What is the x -coordinate of the point with the positive x -coordinate?

Student-produced response: enter your answer.

13 HARD · MODULE 2

In the equation $2x^2 - 8x + c = 0$, c is a constant. If the equation has exactly one real solution, what is the value of c ?

Student-produced response: enter your answer.

14 HARD · MODULE 2

The function f is defined by $f(x) = (x - 3)(x + k)$, where k is a constant. The vertex of the graph of $y = f(x)$ has x -coordinate 1. What is the value of k ?

- A** -1 **B** 1
C 2 **D** 5

15 HARD · MODULE 2

A population is modeled by $P(t) = 800(1.5)^{t/4}$, where t is the time in years. Which statement is true?

- A** It increases by 50% every 4 years.
B It increases by 50% every year.
C It increases by 150% every 4 years.
D It increases by 6% every year.

LEARN FROM EVERY MISTAKE

Before reading the solutions, mark each question you missed with the reason. Over time the pattern tells you exactly what to practice.

Reason	What it means	What to do
Concept	I did not know the rule or idea.	Re-read the section.
Method	I knew the idea but chose a slow or wrong approach.	Study the worked examples.
Misread	I answered a different question (for example x instead of the sum of the solutions).	Underline what is asked.
Careless	An arithmetic or sign slip.	Check with Desmos.

7 Full solutions**1 c**

$$(x+5)(x-2) = x^2 - 2x + 5x - 10 = x^2 + 3x - 10.$$

2 -7

$x = 7$ or $x = -7$; the condition $x < 0$ leaves -7 .

3 5

$f(-2) = 2(-2)^2 - 3 = 2(4) - 3 = 5$. Square first:
 $(-2)^2 = 4$, not -4 .

4 B

Add the exponents: $x^{1/2+3/2} = x^{4/2} = x^2$.

5 B

Doubling means multiplying by 2 each hour, starting from 500: $500(2)^h$. Choice A adds 2 each hour, which is linear.

6 D

$x - 4 = 0$ gives $x = 4$; $2x + 6 = 0$ gives $x = -3$. Choice C forgets to divide by 2.

7 10

$(x-3)(x-7) = 0$, so the solutions are 3 and 7 and their sum is 10. Faster: the sum is $-\frac{b}{a} = 10$.

8 -4

$x^2 - 6x + 5 = (x-3)^2 - 4$. A square is never negative, so the least value is -4 , at $x = 3$.

9 c

The factor $0.9 = 1 - 0.10$, so the value keeps 90% each year: a 10% decrease. Choice D, 90%, is what remains, not what is lost.

10 c

$$\frac{(x-4)(x+4)}{x+4} = x-4.$$

11 3

Square: $2x+3 = x^2$, so $x^2 - 2x - 3 = 0$ and $(x-3)(x+1) = 0$. Check: $x = 3$ gives $\sqrt{9} = 3$ ✓; $x = -1$ gives $\sqrt{1} = 1 \neq -1$, extraneous.

12 4

$x^2 = 2x + 8$, so $x^2 - 2x - 8 = 0$ and $(x-4)(x+2) = 0$. The positive solution is 4 (the point is $(4, 16)$).

13 8

One real solution means the discriminant is zero: $(-8)^2 - 4(2)(c) = 0$, so $64 - 8c = 0$ and $c = 8$.

14 B

The zeros are 3 and $-k$, and the vertex is halfway between them: $\frac{3+(-k)}{2} = 1$, so $3 - k = 2$ and $k = 1$. Check: $f(x) = (x-3)(x+1) = x^2 - 2x - 3$, vertex at $x = 1$. ✓

15 A

When t goes up by 4, the exponent $\frac{t}{4}$ goes up by 1, so P is multiplied by 1.5: a 50% increase every 4 years. Per year the factor is $1.5^{1/4} \approx 1.107$, about 10.7%, so B and D are false; C would need a factor of 2.5.

Want to go further?

These notes are a sample of how I teach. In my live online SAT Math classes we work through all four units (Algebra, Advanced Math, Problem-Solving & Data Analysis, and Geometry & Trigonometry) in small groups, with full-length practice tests reviewed in class.

Book a free demo class: sat@joelsamuelacademy.in

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