

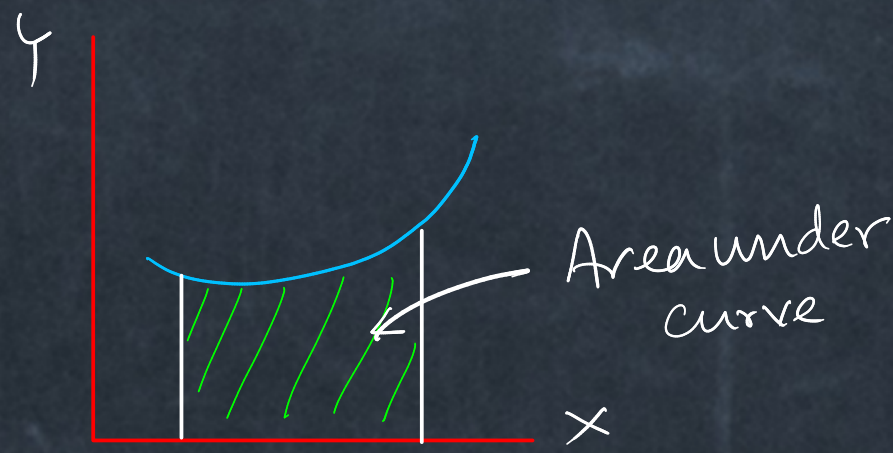
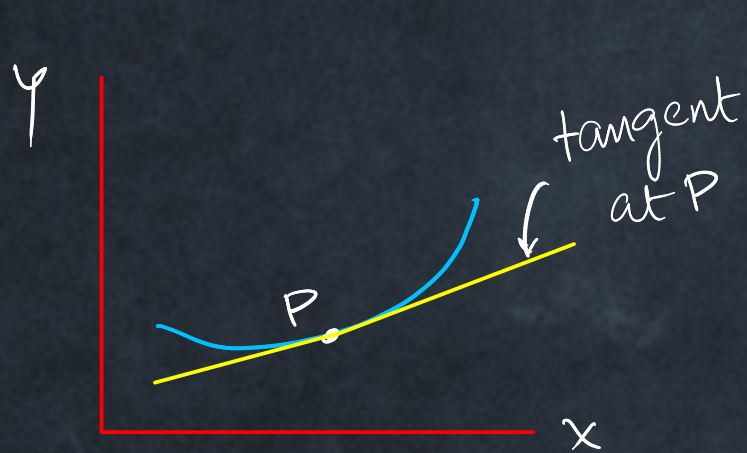
UNIT IV

CALCULUS

12. Limits & Derivatives

LIMITS

- Calculus was created to describe How QUANTITIES CHANGE.
- The 2 basic questions addressed by Calculus are
 - The Instantaneous velocity problem or The tangent problem
 - The Area under a curve.



- The 2 procedures of Calculus {
 - **Differentiation** (for The tangent problem)
 - **Integration** (for the Area problem)

Example 3 Evaluate:

(i) $\lim_{x \rightarrow 1} \frac{x^{15} - 1}{x^{10} - 1}$

(ii) $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

$$(i) \lim_{x \rightarrow 1} \frac{x^{15} - 1}{x^{10} - 1} = \frac{\lim_{x \rightarrow 1} \left(\frac{x^{15} - 1}{x - 1} \right)}{\lim_{x \rightarrow 1} \left(\frac{x^{10} - 1}{x - 1} \right)} = \frac{15(1)^{14}}{10(1)^9} = \frac{15}{10} = \boxed{\frac{3}{2}}$$

(ii) $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$ For $x = 0$ $\frac{f(x)}{g(x)}$ becomes $\frac{0}{0}$ form

Use Rationalization

$$\lim_{x \rightarrow 0} \left[\frac{\sqrt{1+x} - 1}{x} \times \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1} \right] = \lim_{x \rightarrow 0} \left[\frac{x + x - x}{x(\sqrt{1+x} + 1)} \right] = \frac{\cancel{x} \cdot 1}{\cancel{x}(\sqrt{1+x} + 1)}$$

$$\lim_{x \rightarrow 0} \frac{1}{\sqrt{1+x} + 1} = \frac{1}{\sqrt{1+0} + 1} = \boxed{\frac{1}{2}}$$

Additional Problems on limits of Rational functions

$$\textcircled{1} \lim_{x \rightarrow 3} \frac{x^2 - 4x + 3}{x^2 - 2x - 3}$$

For $x = 3$, expression gives $\frac{0}{0}$ form
 $\Rightarrow x - 3$ is common factor

$$\Rightarrow \lim_{x \rightarrow 3} \frac{\cancel{(x-3)}(x-1)}{\cancel{(x-3)}(x+1)} \quad \text{as } x \neq 3$$

$$\Rightarrow \lim_{x \rightarrow 3} \frac{x-1}{x+1} = \frac{3-1}{3+1} = \frac{2}{4} = \frac{1}{2}$$

$$\therefore \lim_{x \rightarrow 3} \frac{x^2 - 4x + 3}{x^2 - 2x - 3} = \frac{1}{2}$$

$$\textcircled{2} \lim_{x \rightarrow 3} \frac{x^4 - 81}{x^2 - 9} = \frac{\lim_{x \rightarrow 3} \left(\frac{x^4 - 3^4}{x - 3} \right)}{\lim_{x \rightarrow 3} \left(\frac{x^2 - 3^2}{x - 3} \right)} = \frac{4(3)^3}{2(3)} = \frac{2 \times 4 \times 27}{2 \times 3} = 18$$

$$\lim_{x \rightarrow 3} \frac{x^4 - 81}{x^2 - 9} = 18$$

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

Derivative as a function

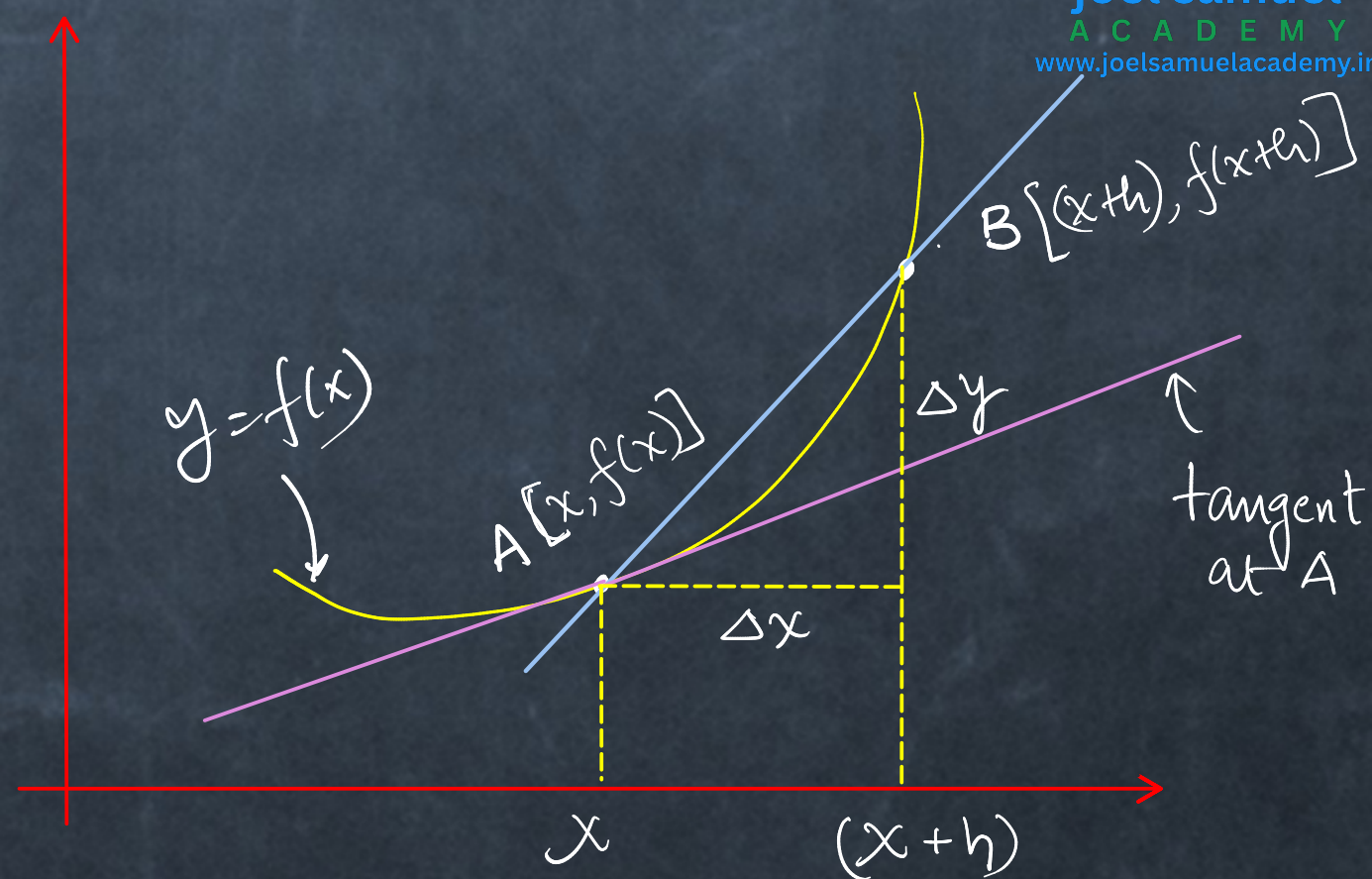
$$f'(x) = \frac{\Delta y}{\Delta x} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{x+h-x}$$

$$f'(x) = \frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

• Also called The first principle of Derivatives

• $f'(x)$ has its domain wherever the above limit exists

$$f'(x) = \frac{df(x)}{dx} = \frac{dy}{dx} = D[f(x)]$$



Example 5 Find the derivative at $x = 2$ of the function $f(x) = 3x$.

$$f(x) = 3 \quad \text{find } f'(2)$$

$$f'(2) = \lim_{h \rightarrow 0} \frac{3(2+h) - 3(2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{6} + 3h - \cancel{6}}{h} = \lim_{h \rightarrow 0} 3 \frac{\cancel{h}}{\cancel{h}} \quad \text{as } h \neq 0$$

$$f'(2) = 3$$

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

Example 6 Find the derivative of the function $f(x) = 2x^2 + 3x - 5$ at $x = -1$. Also prove that $f'(0) + 3f'(-1) = 0$.

$$f(x) = 2x^2 + 3x - 5 \quad \text{find } f'(-1)$$

$$f'(-1) = \lim_{h \rightarrow 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \rightarrow 0} \frac{\{2(-1+h)^2 + 3(-1+h) - 5\} - \{2(-1)^2 + 3(-1) - 5\}}{h}$$

$$f'(-1) = \frac{2(1+h^2-2h) - \cancel{3} + 3h - \cancel{3} - 2 + \cancel{3} + \cancel{5}}{h} = \lim_{h \rightarrow 0} \frac{\cancel{2} + 2h^2 - 4h + 3h - \cancel{2}}{h} = \lim_{h \rightarrow 0} \frac{2h^2 - h}{h}$$

$$\lim_{h \rightarrow 0} \frac{\cancel{h}(2h-1)}{\cancel{h}} \quad \text{as } h \neq 0 \Rightarrow \lim_{h \rightarrow 0} 2h - 1 = -1$$

$$f'(-1) = -1$$

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$