

UNIT II

ALGEBRA

- 4. Complex Numbers
- 5. Linear Inequalities
- 6. Permutations & Combinations
- 7. Binomial Theorem
- 8. Sequences & Series

COMPLEX NUMBERS

Iota(i)

• $i = \sqrt{-1}$; i is called iota.

• To solve $x^2 + 1 = 0$, we use i

$$x^2 = -1$$

$$x = \pm \sqrt{-1} \Rightarrow \boxed{x = \pm i}$$

$$4n \quad 1$$

$$4n+1 \quad i$$

$$4n+2 \quad -1$$

$$4n+3 \quad -i$$

Properties of iota

$$i = i$$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

$$i^5 = i$$

$$i^6 = -1$$

$$i^7 = -i$$

$$i^8 = 1$$

$$i^9 = i \quad \text{-----}$$

$$i^{10} = -1 \quad \text{-----}$$

$$i^{11} = -i \quad \text{-----}$$

$$i^{12} = 1 \quad \text{-----}$$

$$i^{4n+1} = i$$

$$i^{4n+2} = -1$$

$$i^{4n+3} = -i$$

$$i^{4n} = 1$$

Ex: $i^{63} = i^{4(15)+3} = \boxed{-i}$

10. $\left(-2 - \frac{1}{3}i\right)^3$

$$= (-2)^3 - 3(-2)^2\left(\frac{1}{3}i\right) + 3(-2)\left(\frac{1}{3}i\right)^2 - \left(\frac{1}{3}i\right)^3$$

$$i^3 = (-i)$$

$$= -8 - \frac{4}{3}i - \frac{6}{9}i^2 - \frac{2}{27}i^3$$

$$= -8 - 4i - \frac{2}{3}i^2 - \frac{(-i)}{27}$$

$$-\frac{22}{3} - \frac{107}{27}i$$

$$= -8 + \frac{2}{3} + \frac{2}{27} - 4i$$

$$= \frac{-24 + 2}{3} + \frac{i - 108i}{27} =$$

$$-\frac{22}{3} - \frac{107}{27}i$$

Find the multiplicative inverse of each of the complex numbers given in the Exercises 11 to 13.

11. $4 - 3i$

$$z = 4 - 3i$$

$$\frac{1}{z} = \frac{1}{4-3i} \times \frac{4+3i}{4+3i} = \frac{4+3i}{16-9i^2} = \frac{4+3i}{25} = \boxed{\frac{4}{25} + \frac{3}{25}i}$$

12. $\sqrt{5} + 3i$

$$z = \sqrt{5} + 3i$$

$$\frac{1}{z} = \frac{1}{\sqrt{5}+3i} \times \frac{\sqrt{5}-3i}{\sqrt{5}-3i} = \frac{\sqrt{5}-3i}{5-9i^2} = \boxed{\frac{\sqrt{5}}{14} - \frac{3}{14}i}$$

13. $-i$

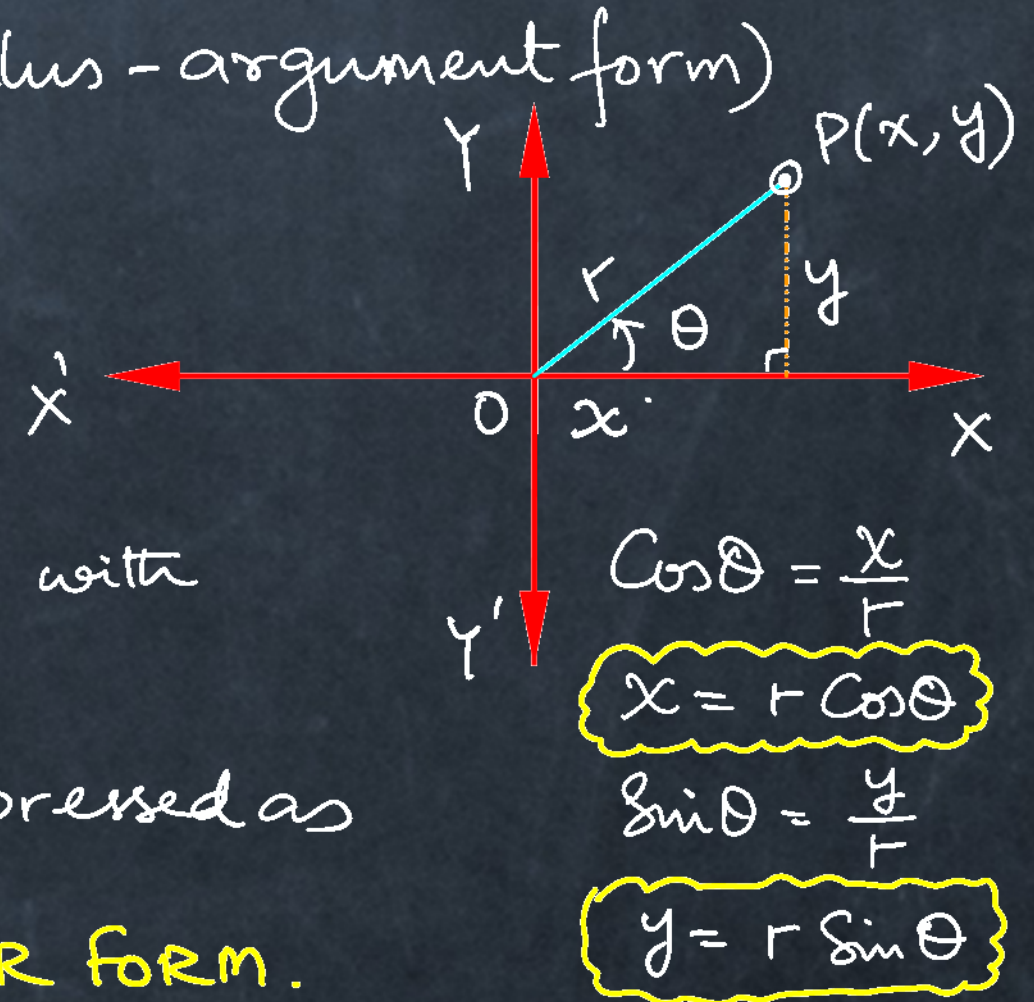
$$z = -i$$

$$\frac{1}{z} = \frac{-1}{-i} \times \frac{i}{i} = \frac{-i}{-i^2} = \frac{-i}{-1} = \boxed{i}$$

Polar Representation of a Complex No

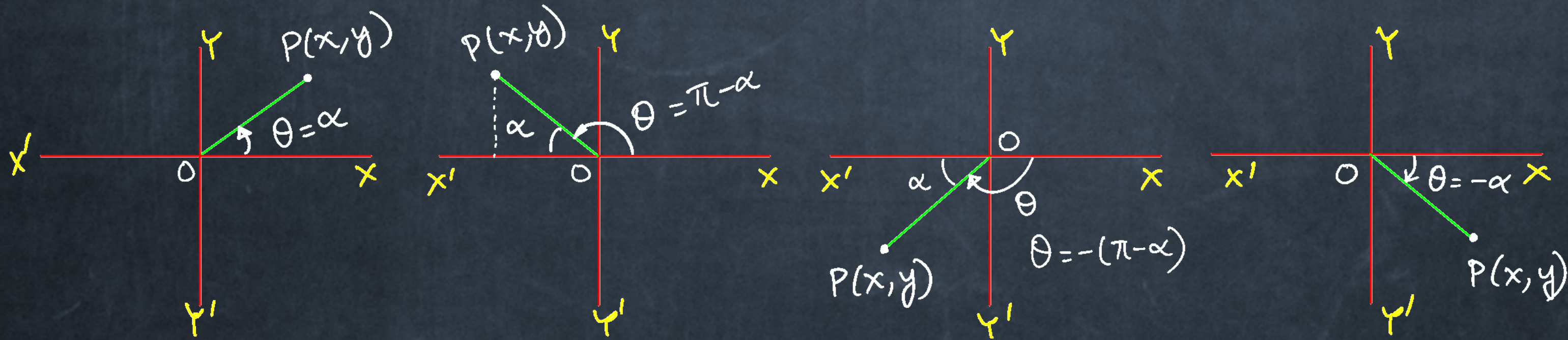
- The point $P(x, y)$ on the Argand plane represents a Complex No. $z = x + iy$
- This $z = x + iy$ can also be represented by another form called **Polar form** (modulus - argument form)
 - modulus r
 - argument $\theta \rightarrow \text{arg } z$
- Here $r = |z| = \sqrt{x^2 + y^2}$
- $\theta = \text{arg } z$ is the angle made by the modulus with the direction of x -axis.
- The Complex No $z = x + iy$ can now be expressed as

$$z = r(\cos \theta + i \sin \theta) \text{ — THE POLAR FORM.}$$



- Generally $\text{arg } z$ refers to all possible θ locating $z = x + iy$.
 ie $\text{arg } z = 2n\pi \pm \theta ; n \in \mathbb{Z}$
 $\hookrightarrow$ Infinite values
- However the Principal argument restricts θ in the interval $(-\pi, \pi]$ ie $-\pi < \theta \leq \pi$

How to determine $\arg z$ ie θ

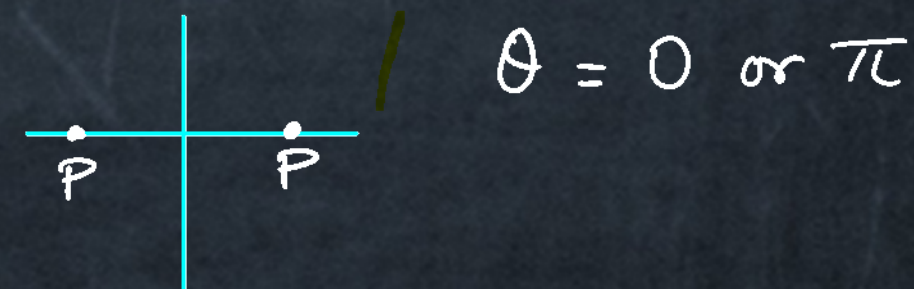


Step 1 $\alpha = \tan^{-1} \left| \frac{y}{x} \right|$

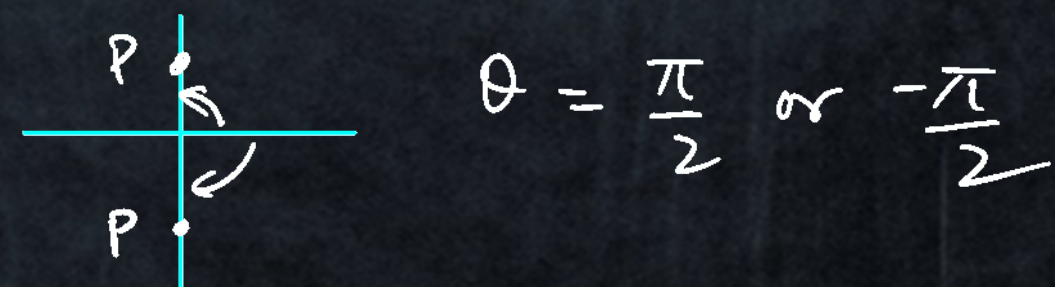
Step 2

Quadrant	$\arg z$ or θ
I	$\theta = \alpha$
II	$\theta = \pi - \alpha$
III	$\theta = -(\pi - \alpha)$
IV	$\theta = -\alpha$

2 Special Cases
I. When $z = x + 0i$



II. When $z = 0 + iy$



LINEAR INEQUALITIES

Linear Inequalities ($>$, $\geq$, $<$, $\leq$)

RULES

ADD/SUBTRACT : $\rightarrow$ Inequality remains unchanged whether (+ve) or (-ve) number is added or subtracted

Ex: $5 > 2$ $5 > 2$
 $5+2 > 2+2$ $5-1 > 2-1$

MULTIPLY/DIVIDE $\rightarrow$ Case I: Multiply or Divide by (+ve) No.
 $\Downarrow$
Inequality remains unchanged

Ex: $5 > 2$ $5 > 2$
 $5(2) > 2(2)$ $\frac{5}{2} > \frac{2}{2}$

Ex. $5 > 2$ $5 > 2$
 $5(-1) < 2(-1)$ $\frac{5}{(-1)} < \frac{2}{(-1)}$

Case II: Multiply or Divide by (-ve) No. *
 $\Downarrow$
Inequality gets reversed

15. $\frac{x}{4} < \frac{(5x-2)}{3} - \frac{(7x-3)}{5}$

$$\frac{x}{4} < \frac{5(5x-2) - 3(7x-3)}{15}$$

$$\frac{x}{4} < \frac{25x - 10 - 21x + 9}{15} \Rightarrow \frac{x}{4} < \frac{4x - 1}{15} \Rightarrow 15x < 16x - 4$$

$$\Rightarrow 4 < x$$

$$x > 4 \Rightarrow x \in (4, \infty)$$

16. $\frac{(2x-1)}{3} \geq \frac{(3x-2)}{4} - \frac{(2-x)}{5}$

$$\frac{2x-1}{3} \geq \frac{5(3x-2) - 4(2-x)}{20}$$

$$\frac{2x-1}{3} \geq \frac{15x - 10 - 8 + 4x}{20} \Rightarrow \frac{2x-1}{3} \geq \frac{19x-18}{20}$$

$$\Rightarrow 20(2x-1) \geq 3(19x-18) \Rightarrow 40x - 20 \geq 57x - 54$$

$$34 \geq 17x \Rightarrow 2 \geq x$$

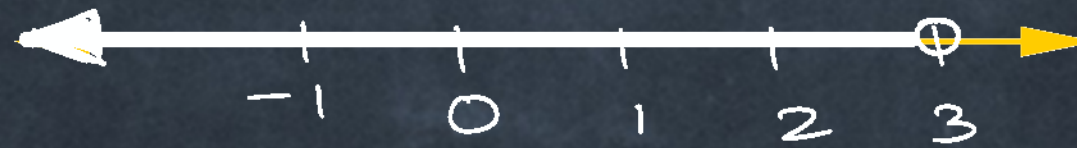
$$x \leq 2 \Rightarrow x \in (-\infty, 2]$$

Solve the inequalities in Exercises 17 to 20 and show the graph of the solution in each case on number line

17. $3x - 2 < 2x + 1$

$$3x - 2 < 2x + 1$$

$$x < 3 \Rightarrow x \in (-\infty, 3)$$

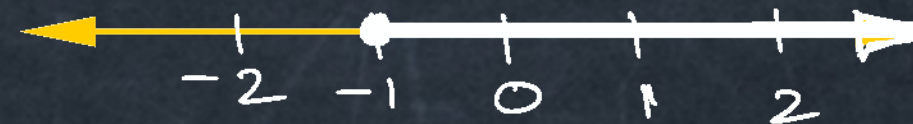


18. $5x - 3 \geq 3x - 5$

$$5x - 3 \geq 3x - 5$$

$$2x \geq -2$$

$$x \geq -1 \Rightarrow x \in [-1, \infty)$$



19. $3(1 - x) < 2(x + 4)$

$$3(1 - x) < 2(x + 4)$$

$$3 - 3x < 2x + 8$$

$$-5 < 5x$$

$$x > -1 \Rightarrow x \in (-1, \infty)$$



Solution of System of Linear Inequalities in Two Variables

EXERCISE 6.3

Solve the following system of inequalities graphically:

1. $x \geq 3, y \geq 2$ $x=3, y=2$

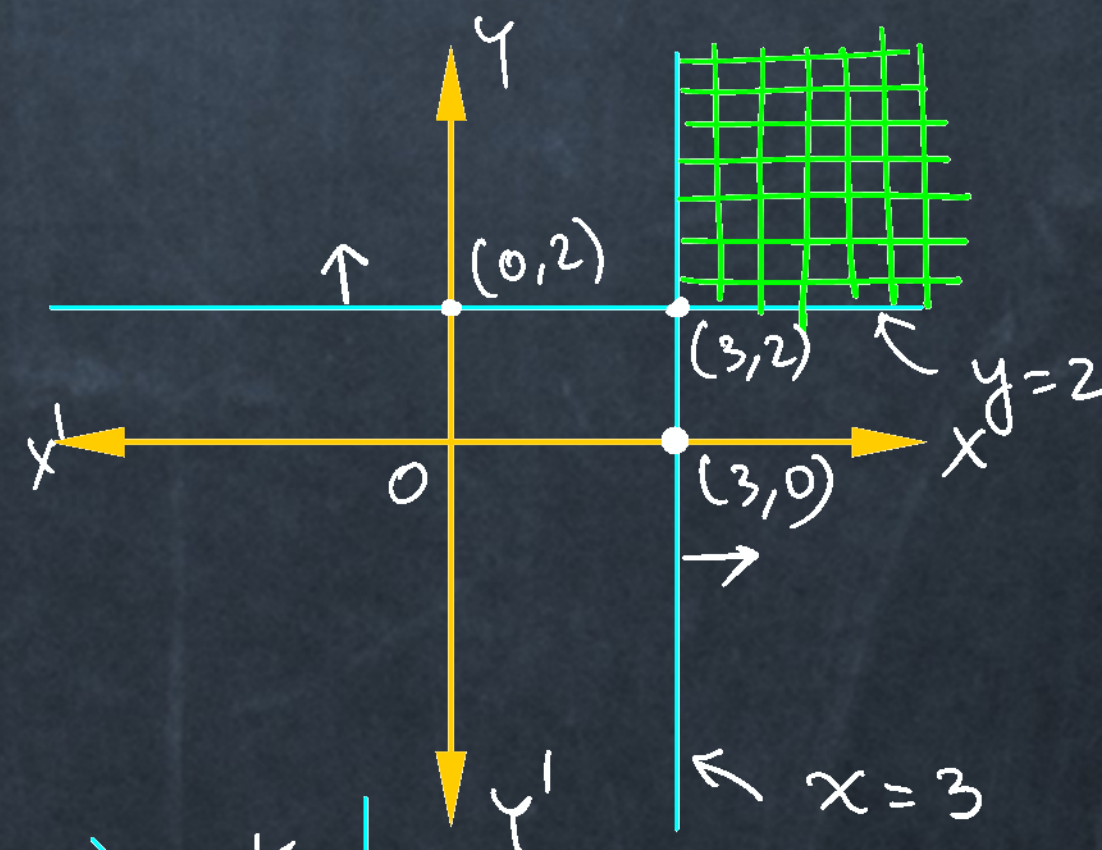
Try $(0,0)$ for

$$x \geq 3$$

$$0 \geq 3 \text{ False}$$

$$y \geq 2$$

$$0 \geq 2 \text{ False}$$



2. $3x + 2y \leq 12, x \geq 1, y \geq 2$

$$3x + 2y \leq 12$$

$$(0,6) (4,0)$$

Try $(0,0)$

$$0 \leq 12$$

True

$$x \geq 1$$

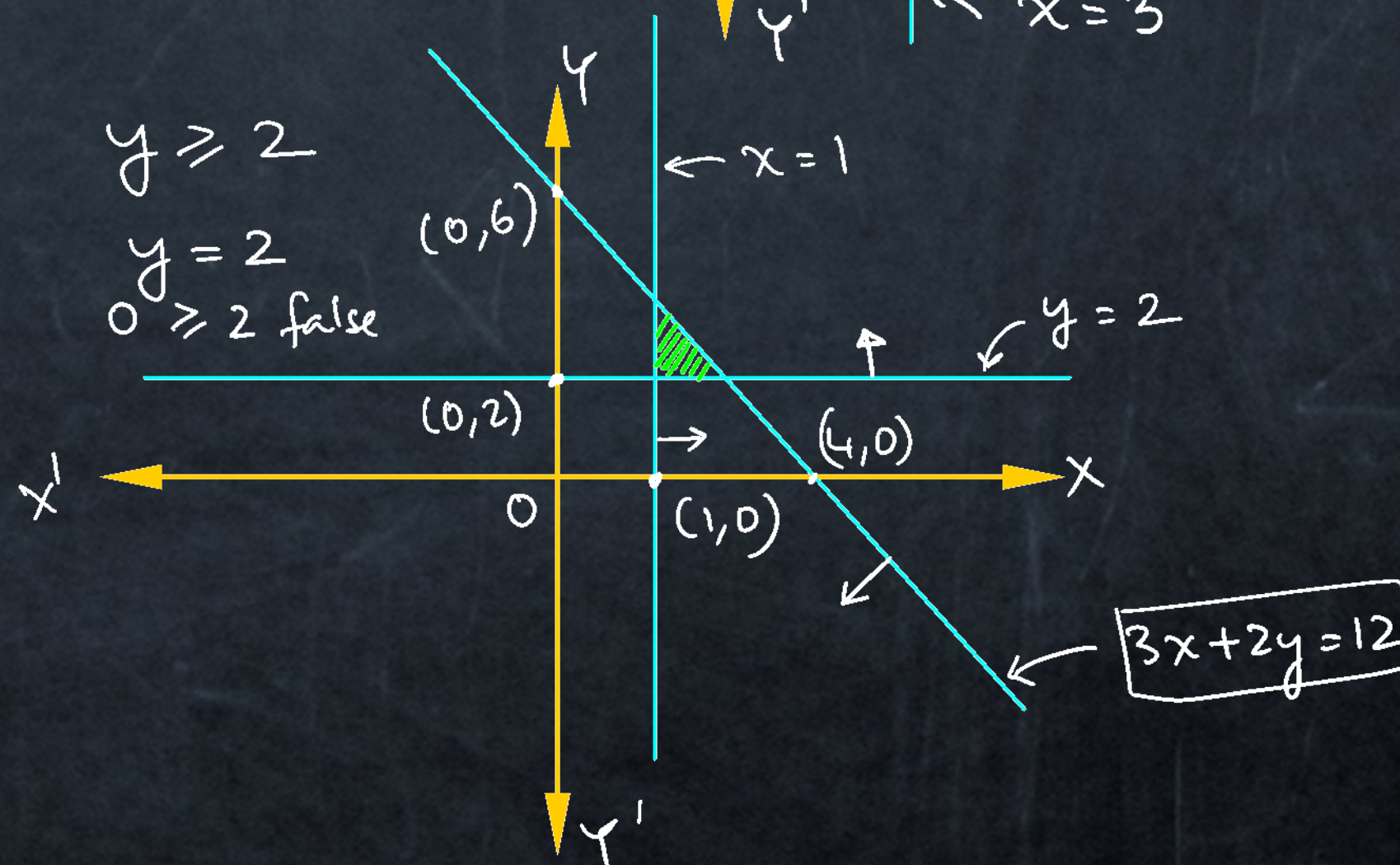
$$x=1$$

$$0 \geq 1$$

$$\text{false}$$

$$y \geq 2$$

$$y=2$$
$$0 \geq 2 \text{ false}$$



7. $2x + y \geq 8, x + 2y \geq 10$

$$2x + y \geq 8$$

$$x + 2y \geq 10$$

$$(0, 8) (4, 0)$$

$$(0, 5) (10, 0)$$

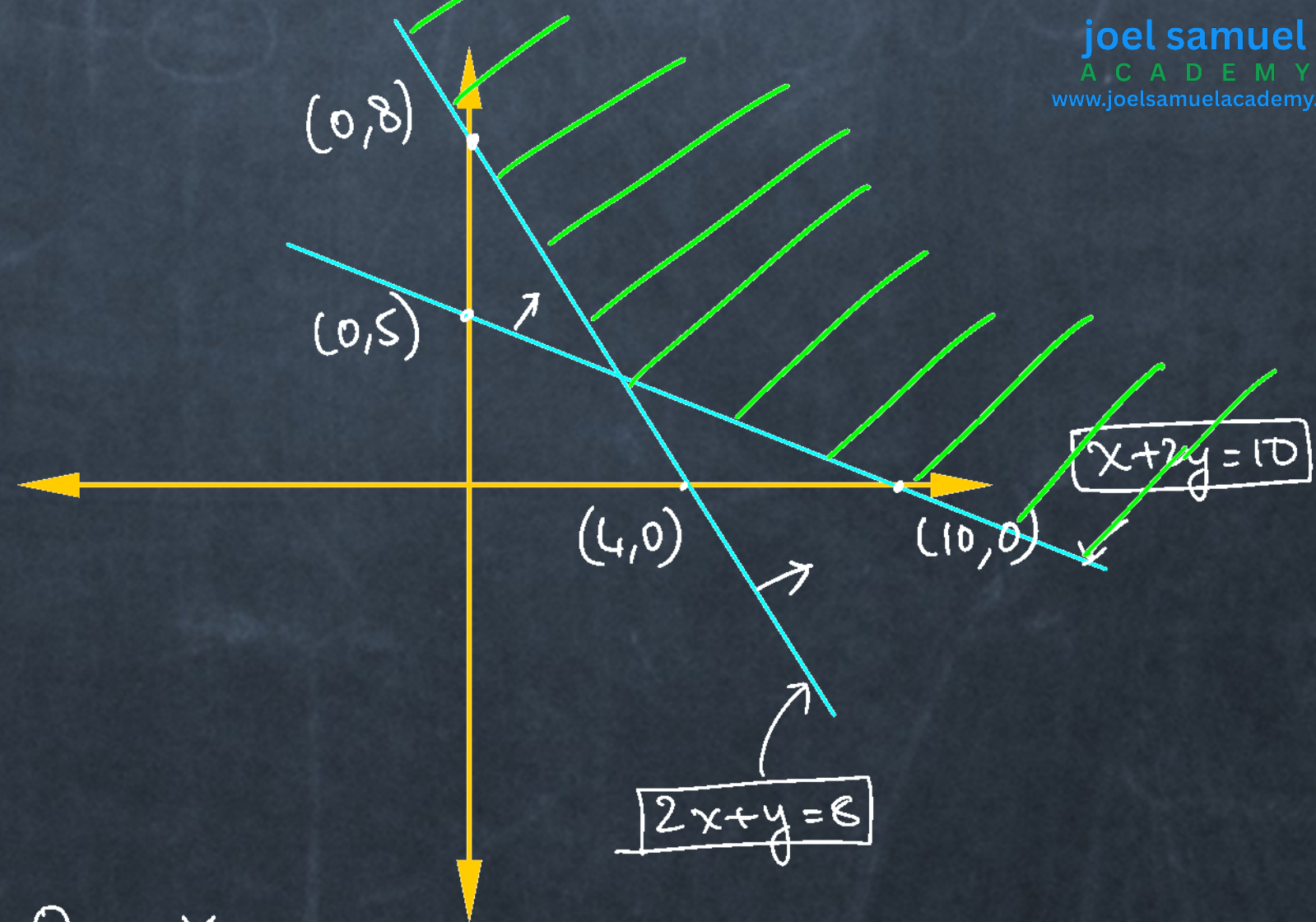
$$(0, 0)$$

$$0 \geq 8$$

false

$$0 \geq 10$$

false



10. $3x + 4y \leq 60, x + 3y \leq 30, x \geq 0, y \geq 0$

$$3x + 4y = 60$$

$$x + 3y = 30$$

$$x = 0$$

$$y = 0$$

$$(0, 15) (20, 0)$$

$$(0, 10) (30, 0)$$

$$0 \leq 60$$

True

$$0 \leq 30$$

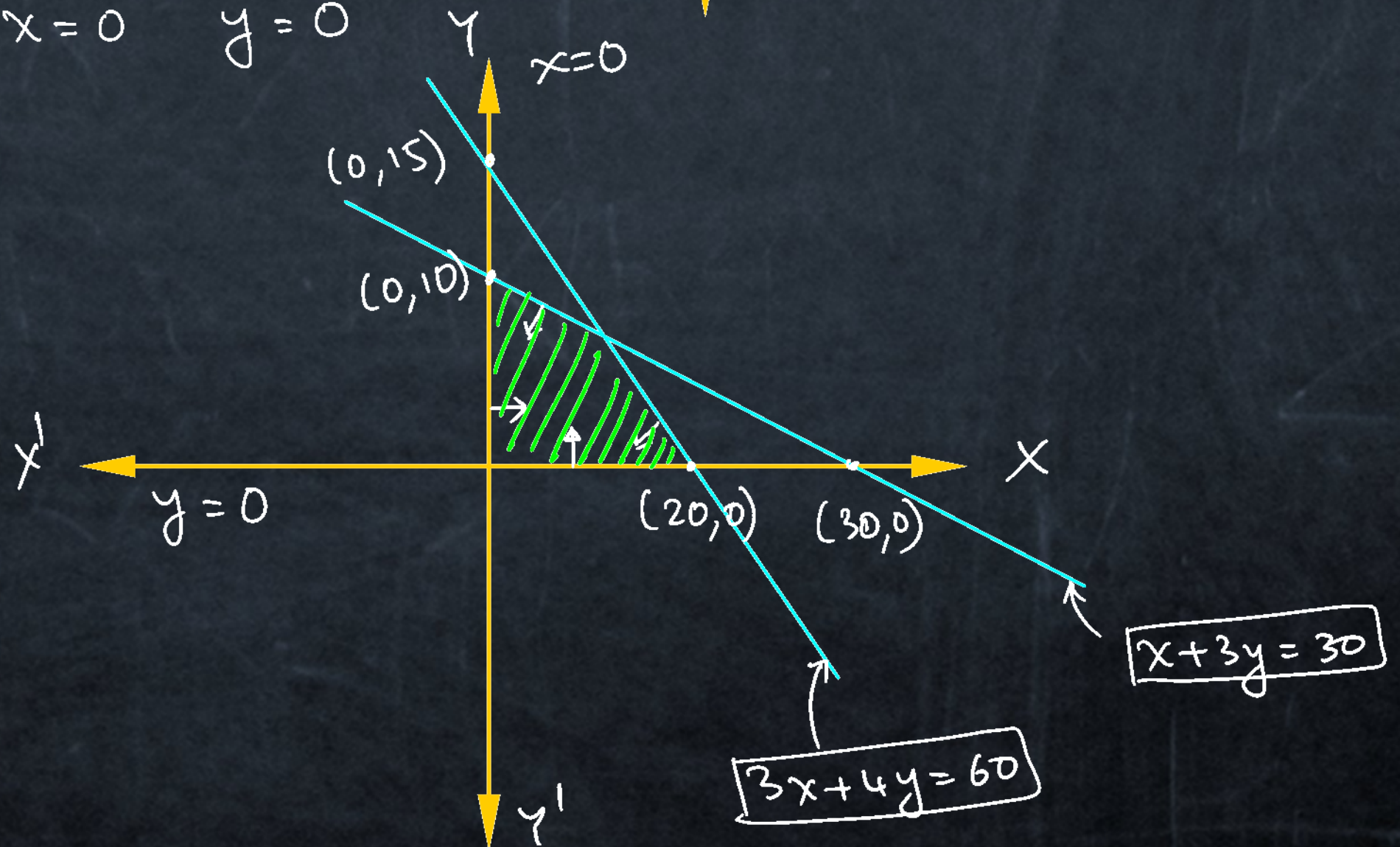
True

For $x \geq 0$ Try $(20, 0)$

$$20 \geq 0$$
 True

For $y \geq 0$ Try $(0, 10)$

$$10 \geq 0$$
 True

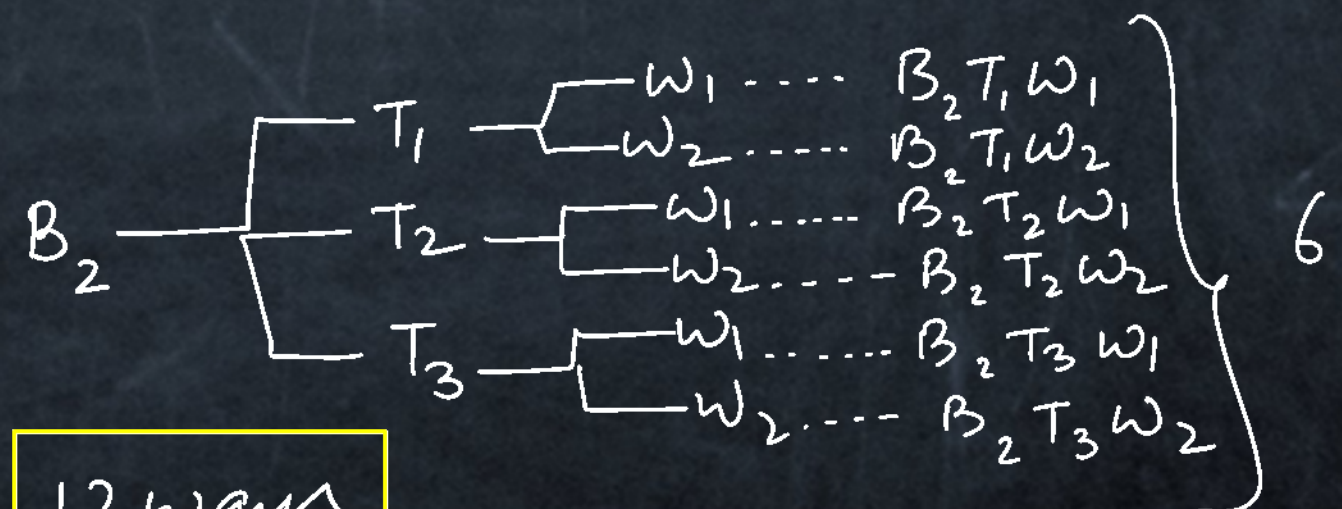
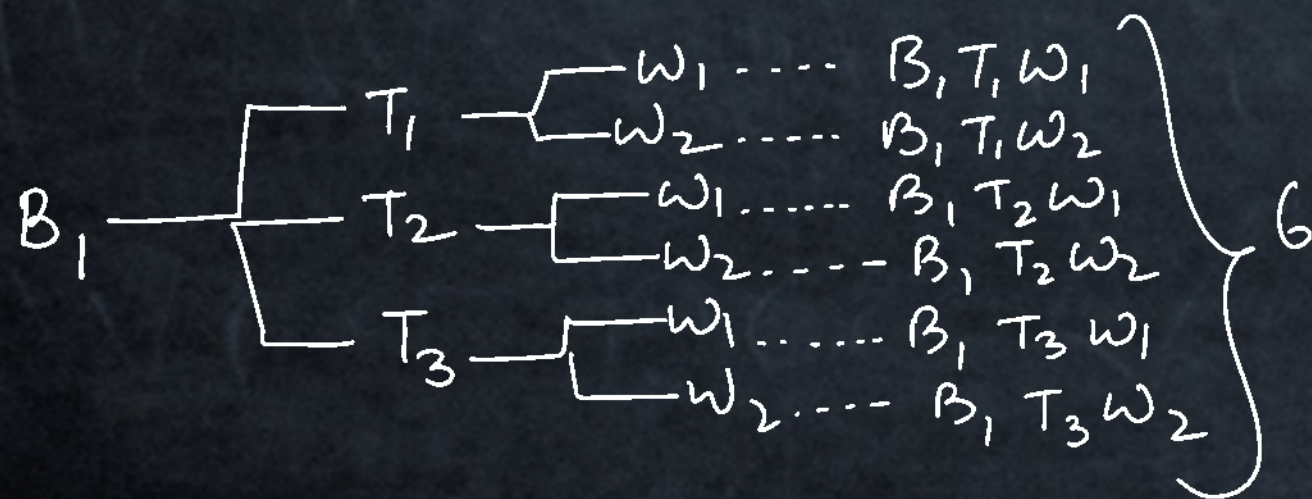


PERMUTATIONS AND COMBINATIONS

I.) A student has 3 pants & 2 shirts. In how many ways can the student dress up?



II.) Sabnam has 2 school bags (B), 3 tiffin boxes (T) & 2 water bottles (W). In how many ways can she carry one of each to school?



$\therefore$ 12 ways

Permutations → Arrangements

Case I - No. of permutations of n objects (all different) taking r at a time where repetition is allowed.



∴ No. of permutations = n^r

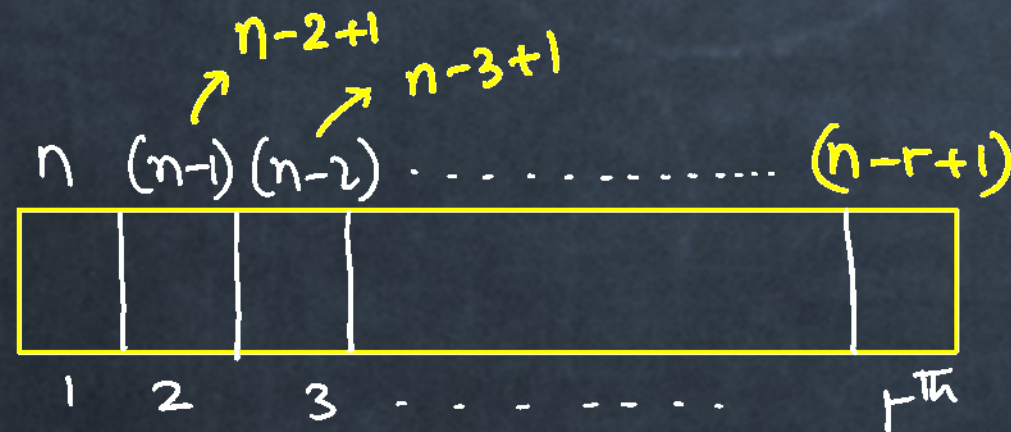
Ex: Arrange letters of the word CAMP taking

(i) 4 letters → $n=4, r=4, \therefore n^r = 4^4 = 256 \text{ ways}$

(ii) 3 letters → $n=4, r=3, \therefore n^r = 4^3 = 64 \text{ ways}$

Repetition is allowed.

Case II : No. of permutations of n objects (all different) taken ' r ' at a time where repetitions is NOT allowed.



$$\text{No. of permutations} = n(n-1)(n-2)\dots(n-r+1)$$

Simplifying we get : $\Rightarrow \frac{n(n-1)(n-2)\dots(n-r+1) [(n-r)(n-r-1)\dots 3 \cdot 2 \cdot 1]}{[(n-r)(n-r-1)\dots 3 \cdot 2 \cdot 1]}$

* * *

$${}_nP_r = \frac{n!}{(n-r)!}$$

$\Rightarrow$ No. of permutations of ' n ' distinct objects taken r at a time without repetition; $(0 \leq r \leq n)$

Note

• $r=0 \Rightarrow {}_nP_0 = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1 \Rightarrow \boxed{{}_nP_0 = 1}$

• $r=1 \Rightarrow {}_nP_1 = \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n \Rightarrow \boxed{{}_nP_1 = n}$

• $r=n \Rightarrow {}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! \Rightarrow \boxed{{}_nP_n = n!}$

10. In how many of the distinct permutations of the letters in MISSISSIPPI do the four I's not come together?

Permutations with I's not together = Total permutations - Permutations with I's together

MISSISSIPPI

M | PP | SSSS | IIII
p=2 q=4 r=4

$$\text{Total permutations} = \frac{11!}{2! 4! 4!} = 34650$$

Permutations with I's together ; IIII + 7 others $\Rightarrow n = 8$
↓
1 block letter

$$\text{Total ways} = \frac{8!}{2! 4!}$$

$$\text{Internal ways in block} = \frac{4!}{4!} = 1$$

$$\therefore \text{Total} = \frac{8!}{2! 4!} \times 1 = 840$$

$$\text{Permutations with I's not together} = 34650 - 840 = \boxed{33810}$$

P | E | R | M | U | TT | A | I | O | N | S

$$\therefore \text{No. of such words} = \frac{10!}{2!} = 1814400$$

AEIOU + 7 other letters $\Rightarrow n = 8$
1 block letter

$$\text{No.} = \frac{8!}{2!} \times {}^5P_5 = 2419200 \text{ words}$$

A horizontal bar divided into 12 equal segments, labeled 1 through 12 below it.

--P-----S--
S-----P

No. of ways for P---S = 7
 " " to interchange P & S = 2

BINOMIAL THEOREM

Binomial Theorem : We use it to expand a binomial raised to any power ie $(a+b)^n$

$$\begin{array}{l}
 (a+b)^0 \\
 (a+b)^1 \\
 (a+b)^2 \\
 (a+b)^3 \\
 \vdots \\
 (a+b)^4 =
 \end{array}
 \begin{array}{c}
 1 \\
 a + b \\
 a^2 + 2ab + b^2 \\
 a^3 + 3a^2b + 3ab^2 + b^3 \\
 a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
 \end{array}$$

Observations

① No. of terms in any expansion of a binomial = (Index + 1)

Ex: $(a+b)^2$ gives 3 terms, $(a+b)^4$ gives 5 terms

$(a+b)^n$ gives $(n+1)$ terms

② In any expansion, power of 'a' decreases while that of 'b' increases

Using binomial theorem, evaluate each of the following:

6. $(96)^3 = (100-4)^3$ By Binomial expansion

$$= {}^3C_0 (100)^3 - {}^3C_1 (100)^2 (4) + {}^3C_2 (100)(4)^2 - {}^3C_3 (4)^3$$

$$= 10^6 - 12 \times 10^4 + 4800 - 64$$

$$= \boxed{884736}$$

$$\begin{array}{r} 1000000 \\ - 120000 \\ \hline 880000 \\ + 4800 \\ \hline 884800 \\ - 64 \\ \hline 884736 \end{array}$$

7. $(102)^5 = (100+2)^5$ By Binomial Theorem

$$= {}^5C_0 (100)^5 + {}^5C_1 (100)^4 (2) + {}^5C_2 (100)^3 (2)^2 + {}^5C_3 (100)^2 (2)^3 + {}^5C_4 (100)(2)^4 + {}^5C_5 (2)^5$$

$$= 10^{10} + 10^9 + 4 \times 10^7 + 8 \times 10^5 + 8000 + 32$$

$$= \boxed{11040808032}$$

$$\begin{array}{r} 10000000000 \\ + 1000000000 \\ + 400000000 \\ + 800000 \\ + 8000 \\ + 32 \\ \hline 11040808032 \end{array}$$

8. $(101)^4 = (100+1)^4$ By Binomial Theorem

$$= {}^4C_0(100)^4 + {}^4C_1(100)^3 + {}^4C_2(100)^2 + {}^4C_3(100) + {}^4C_4$$

$$= 10^8 + 4 \times 10^6 + 6 \times 10^4 + 400 + 1$$

$$= \boxed{104060401}$$

5.3
✓

$$\begin{array}{r} 1000000000 \\ 40000000 \\ 60000 \\ 401 \\ \hline 104060401 \end{array}$$

9. $(99)^5 = (100-1)^5$ By Binomial Theorem

$$= {}^5C_0(100)^5 - {}^5C_1(100)^4 + {}^5C_2(100)^3 - {}^5C_3(100)^2 + {}^5C_4(100) - {}^5C_5$$

$$= 10^{10} - 5 \times 10^8 + 10^7 - 10^5 + 500 - 1$$

$$= \boxed{9509900499}$$

$$\begin{array}{r} 5000000000 \\ 100000 \\ \hline 5001000000 \end{array}$$

(-)

$$\begin{array}{r} 10000000000000 \\ 10000000000 \\ 500 \\ \hline 10010000500 \\ \hline 5001000000 \\ \hline 9509900500 \\ \hline 1 \\ \hline 9509900499 \end{array}$$

$$2) \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{1}{6} \Rightarrow \frac{\frac{n!}{(n-r)! r!}}{\frac{n!}{(n-r-1)! (r+1)!}} = \frac{1}{6}$$

$$\frac{\cancel{n!}}{(n-r)(\cancel{n-r-1})! \cancel{r!}} \times \frac{(\cancel{n-r-1})! (r+1) \cancel{r!}}{\cancel{n!}} = \frac{1}{6}$$

$$6(r+1) = n-r \Rightarrow 6r+6 = n-r \Rightarrow 7r-n = -6 \text{ --- (ii)}$$

Subtract (i) from (ii)

$$\begin{array}{r} 7r - \cancel{n} = -6 \\ \Leftrightarrow 8r - \cancel{n} = \cancel{n} - 1 \\ \hline -r = -7 \text{ or } r = 7 \end{array}$$

$$\text{From } 7r-n = -6 \Rightarrow 7(7)-n = -6 \Rightarrow n = 55$$

$$\therefore \boxed{n=55}$$

Note: The 3 consecutive terms could also be taken as

$$T_{r-1}, T_r, T_{r+1}$$

EXERCISE 8.2

Find the coefficient of

1. x^5 in $(x + 3)^8$

2. a^5b^7 in $(a - 2b)^{12}$.

① $(x + 3)^8$, coeff. of x^5

Solu

General term

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

For $(x + 3)^8$ $T_{r+1} = {}^8C_r (x)^{8-r} (3)^r$

Compare with $x^5 \Rightarrow 8 - r = 5 \Rightarrow r = 3$

$\therefore$ Coeff. of x^5 will be ${}^8C_3 (3)^3 = \frac{8 \times 7 \times 6}{3 \times 2} \times 27 = 1512$

SEQUENCES & SERIES

Terms & Definitions

Sequence — An ordered collection of objects with an identified 1st term, 2nd term & so on according to a certain given rule (this may or maynot be a mathematical formula)

- A sequence can be

finite

Infinite

 ✓
- The terms of a sequence are denoted by $a_1, a_2, a_3, \dots$ or $T_1, T_2, T_3, \dots$

Note: The subscript denoted the position of the term & is ALWAYS a natural number.

Eg: $a_n = -n$ is a sequence

$$n=1, a_1 = -1$$

$$n=2, a_2 = -2$$

$$n=3, a_3 = -3$$

⋮

∴ The sequence is $-1, -2, -3, \dots$

Obs → Output may or maynot be $\mathbb{N}$
Input HAS to $\mathbb{N}$

A sequence has $\mathbb{N}$ as its domain

8. If the sum of n terms of an A.P. is $(pn + qn^2)$, where p and q are constants, find the common difference.

$$S_n = pn + qn^2$$

$$\frac{n}{2} (2a + (n-1)d)$$

$$n=1, S_1 = p(1) + q(1)^2 \Rightarrow S_1 = p + q \Rightarrow a_1 = p + q$$

$$n=2, S_2 = p(2) + q(2)^2 \Rightarrow S_2 = 2p + 4q \Rightarrow a_1 + a_2 = 2p + 4q$$

$$a_2 = a_1 + a_2 - a_1 = 2p + 4q - p - q$$

$$a_2 = p + 3q$$

$$d = a_2 - a_1 \Rightarrow d = p + 3q - p - q$$

$$d = 2q$$

9. The sums of n terms of two arithmetic progressions are in the ratio $5n + 4 : 9n + 6$. Find the ratio of their 18th terms.

Given: $\frac{S_n}{S'_n} = \frac{5n+4}{9n+6}$

Find $\frac{a_{18}}{a'_{18}} = \frac{a_1 + 17d_1}{a_2 + 17d_2}$

$$\Rightarrow \frac{\frac{n}{2} [2a_1 + (n-1)d_1]}{\frac{n}{2} [2a_2 + (n-1)d_2]} = \frac{5n+4}{9n+6} \Rightarrow \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{5n+4}{9n+6} \quad \text{---(i)}$$

$$\frac{a_{18}}{a'_{18}} = \frac{a_1 + 17d_1}{a_2 + 17d_2} = \frac{2a_1 + 34d_1}{2a_2 + 34d_2} = \frac{2a_1 + (35-1)d_1}{2a_2 + (35-1)d_2} = \frac{5(35)+4}{9(35)+6} = \frac{179}{321}$$

$$\frac{a_{18}}{a'_{18}} = \frac{179}{321} \Rightarrow \boxed{a_{18} : a'_{18} = 179 : 321}$$

(c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots$ is $\frac{1}{19683}$?

$$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots$$

$$a = \frac{1}{3}, r = \frac{1}{3}$$

$$a_n = \frac{1}{19683}$$

$$a_n = ar^{n-1}$$

$$\Rightarrow \left(\frac{1}{3}\right) \left(\frac{1}{3}\right)^{n-1} = \frac{1}{19683}$$

$$\frac{1}{3^3} = \frac{1}{3^9} \Rightarrow n = 9$$

$$\begin{array}{r} 3 \overline{) 19683} \\ 3 \overline{) 6561} \\ 3 \overline{) 2187} \\ 3^6 \overline{) 729} \end{array}$$

$\therefore \frac{1}{19683}$ is the 9th term

6. For what values of x , the numbers $-\frac{2}{7}, x, -\frac{7}{2}$ are in G.P.?

For $-\frac{2}{7}, x, -\frac{7}{2}$ to be in G.P.

$$x^2 = \left(-\frac{2}{7}\right) \left(-\frac{7}{2}\right) =$$

$$\Rightarrow x^2 = 1$$

$$x = \pm 1$$

Find the sum to indicated number of terms in each of the geometric progressions in Exercises 7 to 10:

7. 0.15, 0.015, 0.0015, ... 20 terms.

Given G.P. 0.15, 0.015, 0.0015, ...

$$a = 0.15, r = 0.1$$

$$\frac{0.15 \left[(0.1)^{20} - 1 \right]}{0.1 - 1}$$

$$\text{find } S_{20} \quad S_n = \frac{a(1-r^n)}{1-r}$$

$$S_{20} = \frac{0.15 \left[1 - (0.1)^{20} \right]}{1 - 0.1}$$

$$\frac{158}{90186}$$

$$= \frac{0.15}{0.9} \left[1 - (0.1)^{20} \right]$$

$$S_{20} = \frac{1}{6} \left[1 - (0.1)^{20} \right]$$

8. $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \dots n$ terms.

$$a = \sqrt{7}, r = \sqrt{3}$$

$$S_n = \frac{\sqrt{7} \left[(\sqrt{3})^n - 1 \right]}{\sqrt{3} - 1} \Rightarrow \frac{\sqrt{7} \left[(\sqrt{3})^n - 1 \right] (\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)}$$

$$S_n = \frac{\sqrt{7}}{2} \left((\sqrt{3})^n - 1 \right)$$